

Orthogonal Curvilinear Coordinates^{Sp1}

For $\vec{r} = \vec{r}(u_1, u_2, u_3)$,

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \\ &= h_1 du_1 \vec{e}_1 + h_2 du_2 \vec{e}_2 + h_3 du_3 \vec{e}_3 \end{aligned}$$

1. Arc Length & Volume Element

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 \quad dV = h_1 h_2 h_3 du_1 du_2 du_3$$

2. Gradient

$$\vec{\nabla} \Phi = \frac{1}{h_1} \frac{\partial \Phi}{\partial u_1} \vec{e}_1 + \frac{1}{h_2} \frac{\partial \Phi}{\partial u_2} \vec{e}_2 + \frac{1}{h_3} \frac{\partial \Phi}{\partial u_3} \vec{e}_3$$

3. Divergence

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

4. Curl

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \left[h_1 \vec{e}_1 \left\{ \frac{\partial}{\partial u_2} (h_3 A_3) - \frac{\partial}{\partial u_3} (h_2 A_2) \right\} + \text{permutations} \right]$$

5. Laplacian

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u_1} \right) + \text{permutations} \right]$$

Coordinate System	h_1	h_2	h_3	u_1	u_2	u_3
Cartesian	1	1	1	x	y	z
Cylindrical	1	r	1	r	θ	z
Spherical	1	r	$r \sin\theta$	r	θ	ϕ
Frenet-Serret	1	1	$(1+x/\rho)$	x	y	s